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THE NOVEL NUMERICAL SOLUTIONS OF CONFORMABLE FRACTIONAL NAVIER-STOKES EQUATION WITH THE ROBUST METHOD

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ABSTRACT

In this study, new numerical solutions of the conformable fractional Navier-Stokes equation are obtained by the conformable q-Shehu homotopy analysis transform method. The fractional Navier-Stokes equation is an extension of the classical Navier-Stokes equations that incorporates fractional derivatives to account for complex flow behaviors such as memory effects and anomalous diffusion. It is particularly useful in describing fluid dynamics in non-Newtonian fluids, porous media, and other systems with intricate time or spatial dependencies. In addition, two and three-dimensional graphs of the obtained solutions were drawn. We also conduct an error analysis to evaluate the accuracy of the scheme. Computational simulations are performed to validate the accuracy of the upcoming method. This paper presents the conclusions derived from the numerical and graphical analysis.

Keywords: Navier-Stokes equation, Conformable q-Shehu homotopy analysis transform method, Shehu transform.

1 INTRODUCTION

Fractional calculus is a mathematical framework that extends the scope of classical calculus, which deals with derivatives and integrals of integer order, to include derivatives and integrals of arbitrary order [1], [2]. Given that fractional calculus can provide precise descriptions of various nonlinear systems, researchers have begun their research. Fractional differential equations express the influence of nonlocal and inherent material properties [3], [4],

[5]. A cadre of distinguished scientists have conducted extensive research, formulated precise definitions for fractional calculus, and proposed innovative concepts that form the basis of the discipline of fractional calculus studies. Today, fractional calculus is widely used in the development of nonlinear models. A cacophonous environment, financial models derived from chaos theory, optics, and various phenomena have been associated with the concept of fractional calculus [6], [7], [8]. The solution of fractional differential equations is an important tool for characterizing natural nonlinear systems. Researchers obtain exact solutions for fractional differential equations describing nonlinear phenomena by various analytical and numerical methods [9], [10], [48].

An important and useful concept is the conformable fractional derivative. It also helps in understanding the characterization of physical object behavior. Conformable fractional derivative is an effective tool for addressing complex problems. The numerical solutions of differential equations using conformable fractional derivative are comparatively simpler than those using Caputo fractional derivative. This enables the use of conformable fractional derivative, which is advantageous for modeling various physics-related processes. Natural sciences use various fractional models as they can effectively depict real-world problems. A large number of researchers use conformable fractional derivative in various fields. Conformable fractional derivative overcomes certain limitations associated with existing fractional operators [11]-[16], [49], [50].

In 1822, the Navier–Stokes equation, which governs the dynamics of viscous fluid flow, was formulated [17], [18], [19]. The equation combines the energy equation, the continuity equation, and the momentum equation and can be regarded as Newton's second law of motion for fluids. This equation can describe various physical processes such as fluid flow in pipes, ocean currents, air flow, and blood circulation over aircraft wings [20–23]. In 2005, Salem and El-Shahed [24] pioneered the fractional modeling of the Navier–Stokes equations. The authors [24] used the Laplace transform, finite Fourier Sine transform, and finite Hankel transform to generalize the classical Navier–Stokes equations. Kumar et al. [25] solved a nonlinear fractional model of the Navier–Stokes problem analytically by combining the Laplace transform with the homotopic perturbation method. Homotopy analysis method, Ganji et al. [26] and Ragab et al. [27] used it to solve the nonlinear fractional Navier-Stokes equation. Maitama [28] and Momani and Birajdar [29] used the Adomian decomposition method for the numerical calculation of the fractional Navier-Stokes equation. Kumar et al. [30] roughly solved the time-fractional Navier-Stokes problem using the Laplace transform along with the Adomian decomposition method.

On the other hand, the Laplace transform has been used along with the finite Hankel transform to solve the same equation [31].

The q-homotopy analysis transform technique provides various approximate solutions by integrating the q-homotopy analysis method with the Laplace transform [32], [33]. El-Tawil and Huseen [34] proposed q-homotopy analysis method, which is a version of the homotopy analysis method [35], [36] originally described by Liao [37], [38], and the homotopy perturbation method originally described by He [39], [40]. Recent years have witnessed the integration of semi-analytical methods, including the Laplace decomposition technique [41], [42], homotopy perturbation method and the homotopy analysis transform technique, with the Laplace transform algorithm to analyze integer and fractional differential equations modeling real-world problems in scientific and technological fields. In addition, new semi-analytical techniques have been developed, including the conformable fractional Elzaki decomposition method [43] and the conformable q-Shehu analysis transform method [44]. Conformable q-Shehu analysis transform method provides a more generalized form of the series solution by facilitating the convergence of series solutions via the auxiliary parameter h, embedding parameter q, asymptotic parameter n, auxiliary function $H(x,t)$ and initial condition $w_0(x,t)$.

The primary objective of the work is to derive the new numerical solutions for the conformable time-fractional Navier-Stokes equation utilizing conformable q-Shehu homotopy analysis transform method.

This paper is organized as follows: Part 2 elucidates the principles of the conformable fractional derivative, including its essential definitions, notable characteristics, and the core concept of conformable q-Shehu homotopy analysis transform method. The proposed solution is employed to address the conformable time-fractional Navier-Stokes equation in Part 3. The results are delineated in Part 4. The conclusion is located in Part 5.

2 MATERIAL AND METHOD

In the part, the main definitions are introduced.

Definition 2.1. [11], [12] Assume that the function $g:[0,\infty)\to\mathbb{R}$. Thus, conformable fractional derivative of h order α is described by

$$D_\alpha(g)(x)=\lim_{\varepsilon\to 0}\frac{g(x+\varepsilon x^{1-\alpha})-g(x)}{\varepsilon}, \tag{1}$$

for $\forall x>0, 0<\alpha\leq 1$.

Theorem 2.1. [11], [12] Assume that $0 < \alpha \leq 1$ and a, h be α −differentiable at a point $x \in (0, \infty)$. Thus, the following properties exist

(i) $$D_\alpha(c_1 a + c_2 h) = c_1 D_\alpha(a) + c_2 D_\alpha(h) \text{ for } \forall\, c_1, c_2 \in \mathbb{R}, \tag{2}$$

(ii) $$D_\alpha(x^p) = p x^{p-1}, \text{ for } \forall p \in \mathbb{R}, \tag{3}$$

(iii) $$D_\alpha(c) = 0, \text{ for } \forall g(t) = c, \tag{4}$$

(iv) $$D_\alpha(ah) = a D_\alpha(h) + h D_\alpha(a), \tag{5}$$

(v) $$D_\alpha\left(\frac{a}{h}\right) = \frac{h D_\alpha(a) - a D_\alpha(h)}{h^2}. \tag{6}$$

Definition 2.2. [45] Assume that the function $h: [0, \infty) \to \mathbb{R}$. Thus, conformable Shehu transform of $h(t)$ order α is identified as

$$ {}_cS_\alpha[h(t)] = V_\alpha(s; u) = \int_0^\infty exp\left(\frac{-st^\alpha}{u\alpha}\right) h(t) t^{\alpha-1} dt. \tag{7}$$

Definition 2.3. [45] Assume that $\alpha \in (0, 1]$, $h: [0, \infty) \to \mathbb{R}$ be the function. The conformable fractional Shehu transform for the conformable fractional derivative of $h(t)$ is identified by

$$V_\alpha[D_\alpha h(t)](v) = \frac{s}{u} V_\alpha(s; u) - f(0). \tag{8}$$

2.1 The Main Idea of Conformable q-Shehu Homotopy Analysis Transform Method

To clarify the fundamental principle of conformable q-Shehu homotopy analysis transform method **[44]**, it is essential to examine the conformable time-fractional partial differential equations

$$D_t^\alpha u(\xi, \tau) + Au(\xi, \tau) + Hu(\xi, \tau) = \zeta(\xi, \tau), t > 0, n - 1 < \alpha \leq n, \tag{9}$$

where A and H are linear and nonlinear operators, $\zeta(\xi, \tau)$ is a nonhomogeneous function, and D_t^α is conformable fractional derivative of order α.

Applying the conformable fractional Shehu transform to equation (9) and using the initial condition, then, we have

$$\frac{s}{u}\ {}_cS_\alpha[u(\xi, \tau)] - u(x, 0) + \ {}_cS_\alpha[Au(\xi, \tau) + Hu(\xi, \tau)] = \ {}_cS_\alpha[\zeta(\xi, \tau)]. \tag{10}$$

Simplifying the equation (10), equation (20) is derived as

$$ {}_cS_\alpha[u(\xi,\tau)] - \frac{u}{s}u(x,0) + \frac{u}{s}\ {}_cS_\alpha[Au(\xi,\tau) + Hu(\xi,\tau) - \zeta(\xi,\tau)] = 0. \tag{11}$$

Using the homotopy analysis method, the nonlinear operator to the function $\rho(x,t;q)$ is described as

$$ N[\rho(\xi,\tau;q)\ ] = \ {}_cS_\alpha[\rho(\xi,\tau;q)\ ] - \frac{u}{s}\rho(\xi,\tau;q)\ (0^+) + \frac{u}{s}\big(\ {}_cS_\alpha[A\rho(\xi,\tau;q)\big) \tag{12}$$

$(+H\rho(\xi,\tau;q) - \zeta(\xi,\tau)]),$

where $0 \le q \le \frac{1}{n}$.

A homotopy is identified by

$$ (1-nq)\ {}_cS_\alpha[\rho(\xi,\tau;q) - u_0(\xi,\tau)] = hqH^*(\xi,\tau)H[\rho(\xi,\tau;q)], \tag{13}$$

where, $h \neq 0$. For $q = 0$ and $q = \frac{1}{n}$, the outcomes of equation (13) are obtained as

$$ \rho(\xi,\tau;0) = u_0(\xi,\tau), \rho\left(\xi,\tau;\frac{1}{n}\right) = u(\xi,\tau). \tag{14}$$

Therefore, while increasing the value of q from 0 to $1/n$, the solution $\rho(\xi,\tau;q)$ approaches the solution $u(\xi,\tau)$ by beginning at $u_0(\xi,\tau)$. Using Taylor theorem around q and thereafter expanding $\rho(\xi,\tau;q)$, the resulting expression is produced by

$$ \rho(\xi,\tau;q) = u_0(\xi,\tau) + \sum_{\mathrm{m}=1}^{\infty} u_m(x,t)q^m, \tag{15}$$

where,

$$ u_m(\xi,\tau) = \frac{1}{m!}\frac{\partial^m \rho(\xi,\tau;q)}{\partial q^m}|_{q=0}. \tag{16}$$

Equation (15) approaches at $q = \frac{1}{n}$ for the appropriate $u_0(\xi,\tau)$, n and h. Subsequently, the solution for the initial nonlinear equation in the specified format is acquired by

$$ u(\xi,\tau) = u_0(\xi,\tau) + \sum_{m=1}^{\infty} u_{\mathrm{m}}(\xi,\tau)\left(\frac{1}{n}\right)^m. \tag{17}$$

By taking the $m-th$ derivative of the zeroth order deformation equation (13) with respect to q and dividing by the factorial of m, when $q = 0$, we obtain

$$ {}_cS_\alpha[u_m(\xi,\tau) - \chi_m u_{m-1}(\xi,\tau)] = hH^*(\xi,\tau)\mathcal{R}_{*m}(\vec{u}_{m-1}), \tag{18}$$

where the vectors are identified as

$$ \vec{u}_m = \{u_0(\xi,\tau), u_1(\xi,\tau), \dots, u_m(\xi,\tau)\}. \tag{19}$$

Inverse conformable fractional Shehu transform is applied to equation (18), it is obtained as

$$ u_{\mathrm{m}}(\xi,\tau) = \chi_m u_{m-1}(\xi,\tau) + h\left({}_cS_\alpha\right)^{-1}[H^*(\xi,\tau)\mathcal{R}_{*m}(\vec{u}_{m-1})], \tag{20}$$

where

$$ \begin{aligned} \mathcal{R}_{*m}(\vec{u}_{m-1}) = {}& {}_cS_\alpha[u_{m-1}(\xi,\tau)] \\ -\left(1-\frac{\chi_m}{n}\right)\frac{u}{s}u_0(\xi,\tau) + \frac{\mathrm{u}}{s}\,{}_cS_\alpha[Au_{m-1}(\xi,\tau) + H^*{}_{m-1}(\xi,\tau) - \zeta(\xi,\tau)], \end{aligned} \tag{21}$$

and

$$ \chi_m = \begin{cases} 0, & m \le 1, \\ n, & m > 1. \end{cases} \tag{22}$$

Here, $H^*{}_m$ is homotopy polynomial and presented as

$$ H^*{}_m = \frac{1}{m!}\frac{\partial^m \rho(\xi,\tau;q)}{\partial q^m}|_{q=0} \tag{23}$$

and $\rho(\xi,\tau;q) = \rho_0 + q\rho_1 + q^2\rho_2 + \cdots$.

Utilizing equations (20)-(21), one gets

$$ \begin{aligned} u_{\mathrm{m}}(\xi,\tau) = {}& (\chi_m + h)u_{\mathrm{m-1}}(\xi,\tau) - \left(1-\frac{\chi_m}{n}\right)\frac{u}{s}u_0(\xi,\tau) \\ &+h\left({}_cS_\alpha\right)^{-1}\left[\frac{\mathrm{u}}{s}\left({}_cS_\alpha[Au_{\mathrm{m-1}} + H^*{}_{m-1}(\xi,\tau) - \zeta(\xi,\tau)]\right)\right]. \end{aligned} \tag{24}$$

Using conformable q-Shehu homotopy analysis transform method, then we have

$$ u(\xi,\tau) = \sum_{m=0}^{\infty} u_{\mathrm{m}}(\xi,\tau). \tag{25}$$

2.2 Convergence Analysis

In this section, two theorems related to conformable q-Shehu homotopy analysis transform method are given.

Theorem 2.2.1. [46]. The solution of conformable time-fractional partial differential equations acquired by conformable q-Shehu homotopy analysis transform method is unique for $\forall \varsigma \epsilon (0,1)$, where $\varsigma = (n + \hbar) + \hbar(\epsilon + \rho)\Lambda$.

Theorem 2.2.2. [46]. Assume that Θ is Banach space and $\chi: \Theta \to \Theta$ is the nonlinear mapping. Let the following inequality is provided as

$$\|\boldsymbol{\chi}(\boldsymbol{v}) - \boldsymbol{\chi}(\boldsymbol{\rho})\| \leq \boldsymbol{\gamma}\|\boldsymbol{v} - \boldsymbol{\rho}\|, \forall \boldsymbol{v}, \boldsymbol{\rho} \in \boldsymbol{\Theta}, \tag{26}$$

therefore χ has a fixed point in Banach space [47]. Furthermore, for any arbitrary selection of $m_0, n_0 \in \Theta$, the sequence acquired via the conformable q-Shehu homotopy analysis transform method converges to a fixed point of χ and

$$\|\boldsymbol{v_m} - \boldsymbol{v_n}\| \leq \frac{\boldsymbol{\gamma^n}}{\boldsymbol{1-\gamma}}\|\boldsymbol{v_1} - \boldsymbol{v_0}\|, \forall \boldsymbol{v}, \boldsymbol{\rho} \in \boldsymbol{\Theta}. \tag{27}$$

3 APPLICATION

Consider the nonlinear conformable time-fractional Navier-Stokes equation [32], [33]

$$D_t^{\alpha} w(x,y,t) + w(x,y,t)\frac{\partial w(x,y,t)}{\partial x} + q(x,y,t)\frac{\partial w(x,y,t)}{\partial y} = \rho\left[\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right] + \sigma, 0 < \alpha \leq 1, \tag{28}$$

$$D_t^{\alpha} q(x,y,t) + w(x,y,t)\frac{\partial q(x,y,t)}{\partial x} + q(x,y,t)\frac{\partial q(x,y,t)}{\partial y} = \rho\left[\frac{\partial^2 q(x,y,t)}{\partial x^2} + \frac{\partial^2 q(x,y,t)}{\partial y^2}\right] - \sigma, 0 < \alpha \leq 1, \tag{29}$$

subject to initial conditions

$$w(x,y,0) = -\sin(x+y), \tag{30}$$

$$q(x,y,0) = \sin(x+y). \tag{31}$$

The parameter α in the fractional derivative is a critical physical parameter that describes the memory effect and anomalous diffusion behavior of the system. When $\alpha = 1$, the classical (integer order) Navier-Stokes equation is obtained and the flow exhibits purely local behavior. When $\alpha < 1$, the past states of the system affect the flow dynamics. This is important in systems that contain memory effects, such as viscoelastic flows, fluid motion in porous media, and biological fluids. σ represents the external forces or source/sink term in the system.

Physically, this term can be the result of an external force, heat source, magnetic effect, or chemical reaction acting on the fluid.

Since σ is positive in equation (28) and negative in equation (29), these terms can be considered to have a balanced effect with each other. ρ is used as the coefficient of the viscous diffusion term in the solution expressions. It has a physical meaning as the density of the fluid. High values of ρ indicate that the fluid has a high inertia. Low values of ρ are valid for lighter fluids or gases. In fractional derivative models, ρ can be an important parameter that changes the fractional flow behavior.

Implementing the conformable fractional Shehu transform to equations (28)-(29) and utilizing the initial conditions, they are acquired by

$$\frac{s}{u}\,{}_cS_\alpha[w(x,y,t)] - w(x,y,0) + {}_cS_\alpha\left[w(x,y,t)\frac{\partial w(x,y,t)}{\partial x} + q(x,y,t)\frac{\partial w(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right\} - \sigma\right] = 0, \tag{32}$$

$$\frac{s}{u}\,{}_cS_\alpha[q(x,y,t)] - q(x,y,0) + {}_cS_\alpha\left[w(x,y,t)\frac{\partial q(x,y,t)}{\partial x} + q(x,y,t)\frac{\partial q(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2 q(x,y,t)}{\partial x^2} + \frac{\partial^2 q(x,y,t)}{\partial y^2}\right\} + \sigma\right] = 0. \tag{33}$$

By arranging the equations (32)-(33), equations (34)-(35) are generated as

$${}_cS_\alpha[w(x,y,t)] - \frac{u}{s}w(x,y,0) + \frac{u}{s}\,{}_cS_\alpha\left[w(x,y,t)\frac{\partial w(x,y,t)}{\partial x} + q(x,y,t)\frac{\partial w(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2 w(x,y,t)}{\partial x^2} + \frac{\partial^2 w(x,y,t)}{\partial y^2}\right\} - \sigma\right] = 0, \tag{34}$$

$${}_cS_\alpha[q(x,y,t)] - \frac{u}{s}q(x,y,0) + \frac{u}{s}\,{}_cS_\alpha\left[w(x,y,t)\frac{\partial q(x,y,t)}{\partial x} + q(x,y,t)\frac{\partial q(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2 q(x,y,t)}{\partial x^2} + \frac{\partial^2 q(x,y,t)}{\partial y^2}\right\} + \sigma\right] = 0. \tag{35}$$

Via the homotopy analysis method, the nonlinear operators for real functions $\varphi(x,t;q)$ and $\psi(x,t;q)$ are defined as

$$N^1[\varphi(x,t;q)\,] = {}_cS_\alpha[\varphi(x,t;q)\,] - \frac{u}{s}\varphi(x,t;q)\,(0^+) + \frac{u}{s}\,{}_cS_\alpha[\varphi(x,y,t)\frac{\partial\varphi(x,y,t)}{\partial x} + \psi(x,y,t)\frac{\partial\varphi(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2\varphi(x,y,t)}{\partial x^2} + \frac{\partial^2\varphi(x,y,t)}{\partial y^2}\right\} - \sigma\Big], \tag{36}$$

$$N^2[\psi(x,t;q)\,] = {}_cS_\alpha[\psi(x,t;q)\,] - \frac{u}{s}\psi(x,t;q)\,(0^+) + \frac{u}{s}\,{}_cS_\alpha[\varphi(x,y,t)\frac{\partial\psi(x,y,t)}{\partial x} + \psi(x,y,t)\frac{\partial\psi(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2\psi(x,y,t)}{\partial x^2} + \frac{\partial^2\psi(x,y,t)}{\partial y^2}\right\} + \sigma\Big]. \tag{37}$$

By applying the proposed algorithm, the $m-th$ order deformation equations are identified as

$${}_cS_\alpha[w_m(x,y,t) - k_m w_{m-1}(x,y,t)] = h\mathcal{R}_{1,m}(\vec{w}_{m-1}), \tag{38}$$

$${}_cS_\alpha[q_m(x,y,t) - k_m q_{m-1}(x,y,t)] = h\mathcal{R}_{2,m}(\vec{q}_{m-1}), \tag{39}$$

where

$$\mathcal{R}_{1,m}(\vec{w}_{m-1}) = {}_cS_\alpha[w_{m-1}(x,\mathrm{y},t)] - \left(1-\frac{k_m}{n}\right)\frac{u}{s}w_0(x,\mathrm{y},t) + \frac{u}{s}\,{}_cS_\alpha\left[\sum_{r=0}^{m-1} w_r(x,y,t)\frac{\partial w_{m-1-r}(x,y,t)}{\partial x} + \sum_{r=0}^{m-1} q_r(x,y,t)\frac{\partial w_{m-1-r}(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2 w_{m-1}(x,\mathrm{y},t)}{\partial x^2} + \frac{\partial^2 w_{m-1}(x,\mathrm{y},t)}{\partial y^2}\right\} - \sigma\right], \tag{40}$$

$$\mathcal{R}_{2,m}(\vec{q}_{m-1}) = {}_cS_\alpha[q_{m-1}(x,\mathrm{y},t)] - \left(1-\frac{k_m}{n}\right)\frac{u}{s}q_0(x,\mathrm{y},t) + \frac{u}{s}\,{}_cS_\alpha\left[\sum_{r=0}^{m-1} w_r(x,y,t)\frac{\partial q_{m-1-r}(x,y,t)}{\partial x} + \sum_{r=0}^{m-1} q_r(x,y,t)\frac{\partial q_{m-1-r}(x,y,t)}{\partial y} - \rho\left\{\frac{\partial^2 q_{m-1}(x,\mathrm{y},t)}{\partial x^2} + \frac{\partial^2 q_{m-1}(x,\mathrm{y},t)}{\partial y^2}\right\} + \sigma\right]. \tag{41}$$

When equations (38)-(39) are implemented to the inverse conformable fractional Shehu transform, we have

$$w_m(x,\mathrm{y},t) = k_m w_{m-1}(x,\mathrm{y},t) + h\left({}_cS_\alpha\right)^{-1}\left[\mathcal{R}_{1,m}(\vec{w}_{m-1})\right], \tag{42}$$

$$q_m(x,\mathrm{y},t) = k_m w_{m-1}(x,\mathrm{y},t) + h\left({}_cS_\alpha\right)^{-1}\left[\mathcal{R}_{2,m}(\vec{q}_{m-1})\right]. \tag{43}$$

By the use of initial conditions, it is obtained as

$$w_0(x,y,t) = -\sin(x+y), \tag{44}$$

$$q_0(x,y,t) = \sin(x+y). \tag{45}$$

Putting $m=1, m=2$ in the equations (42)-(43) respectively, equations (46)-(47) are generated as

$$w_1(x,y,t) = -\frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha}, \tag{46}$$

$$q_1(x,y,t) = \frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha}, \tag{47}$$

$$w_2(x,y,t) = (n+h)\left[-\frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha}\right] - \left[\frac{4\rho^2 h^2 \sin(x+y)\, t^{2\alpha}}{2\alpha^2}\right], \tag{48}$$

$$q_2(x,y,t) = (n+h)\left[\frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha}\right] + \left[\frac{4\rho^2 h^2 \sin(x+y)\, t^{2\alpha}}{2\alpha^2}\right]. \tag{49}$$

Thus, the conformable q-Shehu homotopy analysis transform method (Cq-SHATM) solutions of equaitons (28)-(29) are acquired as

$$\begin{aligned} w(x,y,t) &= w_0(x,y,t) + \sum_{m=1}^{\infty} w_m(x,y,t)\left(\frac{1}{n}\right)^m \\ &= -\sin(x+y) - \frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha} + (n+h)\left[-\frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha}\right] \\ &\quad - \left[\frac{4\rho^2 h^2 \sin(x+y)\, t^{2\alpha}}{2\alpha^2}\right], \end{aligned} \tag{50}$$

$$\begin{aligned} q(x,y,t) &= q_0(x,y,t) + \sum_{m=1}^{\infty} q_m(x,y,t)\left(\frac{1}{n}\right)^m \\ &= \sin(x+y) + \frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha} + (n+h)\left[\frac{2\rho h \sin(x+y)\, t^{\alpha}}{\alpha}\right] \\ &\quad + \left[\frac{4\rho^2 h^2 \sin(x+y)\, t^{2\alpha}}{2\alpha^2}\right]. \end{aligned} \tag{51}$$

For $\alpha = 1, h = -1, n = 1$ and $\sigma = 0$, the solutions $\sum_{m=1}^{N} w_m(x,y,t)\left(\frac{1}{n}\right)^m$ and $\sum_{m=1}^{N} q_m(x,y,t)\left(\frac{1}{n}\right)^m$ converge to exact results as $N \to \infty$:

$$w(x,y,t) = -\sin(x+y)\left[1 - \frac{2\rho t}{1!} + \frac{(2\rho t)^2}{2!} + \cdots\right] = -e^{-2\rho t}\, sin(x+y), \tag{52}$$

$$q(x,y,t) = \sin(x+y)\left[1 + \frac{2\rho t}{1!} + \frac{(2\rho t)^2}{2!} + \cdots\right] = e^{-2\rho t}\sin(x+y). \tag{53}$$

Figure 1. shows three dimensional plots of distinct α values at $\rho = 0.5$, $\sigma = 0$, $n = 1$ and $h = -1$ for equation (28).

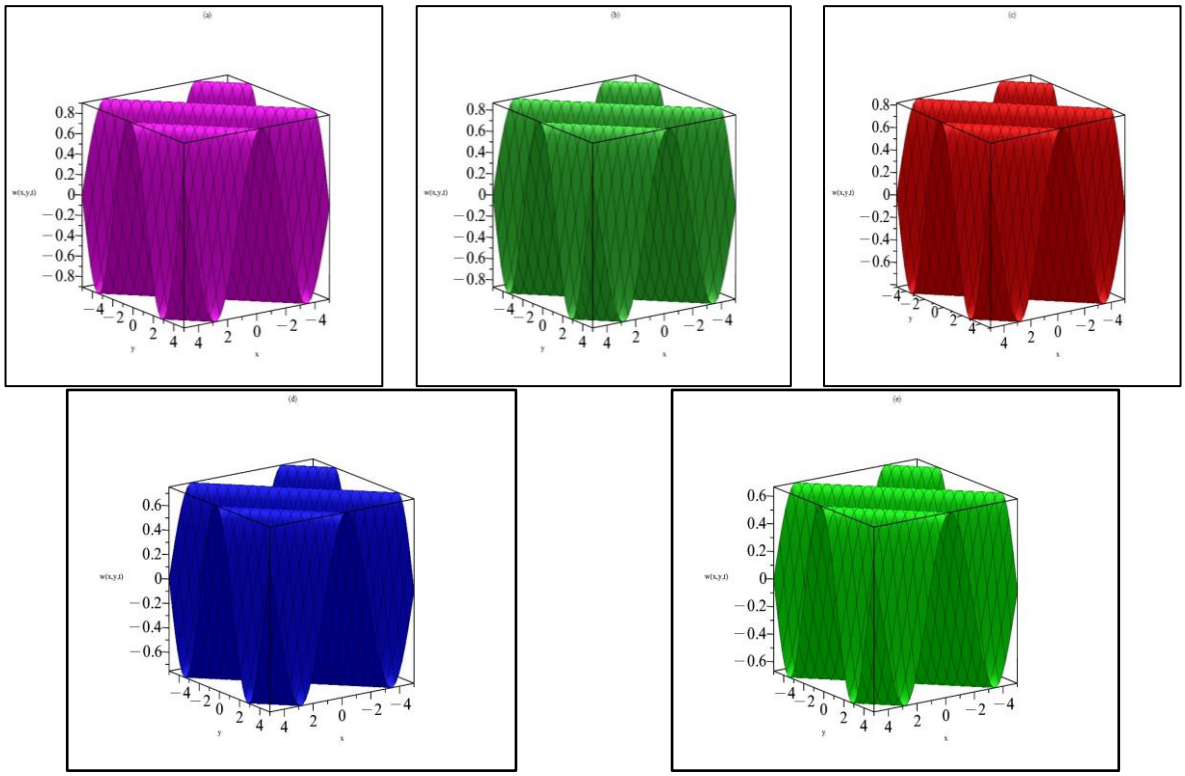


Figure. 1. (a) For $\alpha = 1$, Cq-SHATM solutions (b) For $\alpha = 0.9$, Cq-SHATM solutions (c) For $\alpha = 0.8$, Cq-SHATM solutions (d) For $\alpha = 0.7$, Cq-SHATM solutions (e) For $\alpha = 0.6$, Cq-SHATM solutions at $\rho = 0.5$, $\sigma = 0$, $n = 1$ and $h = -1$ for Equation (28) $h = -1, n = 1$.

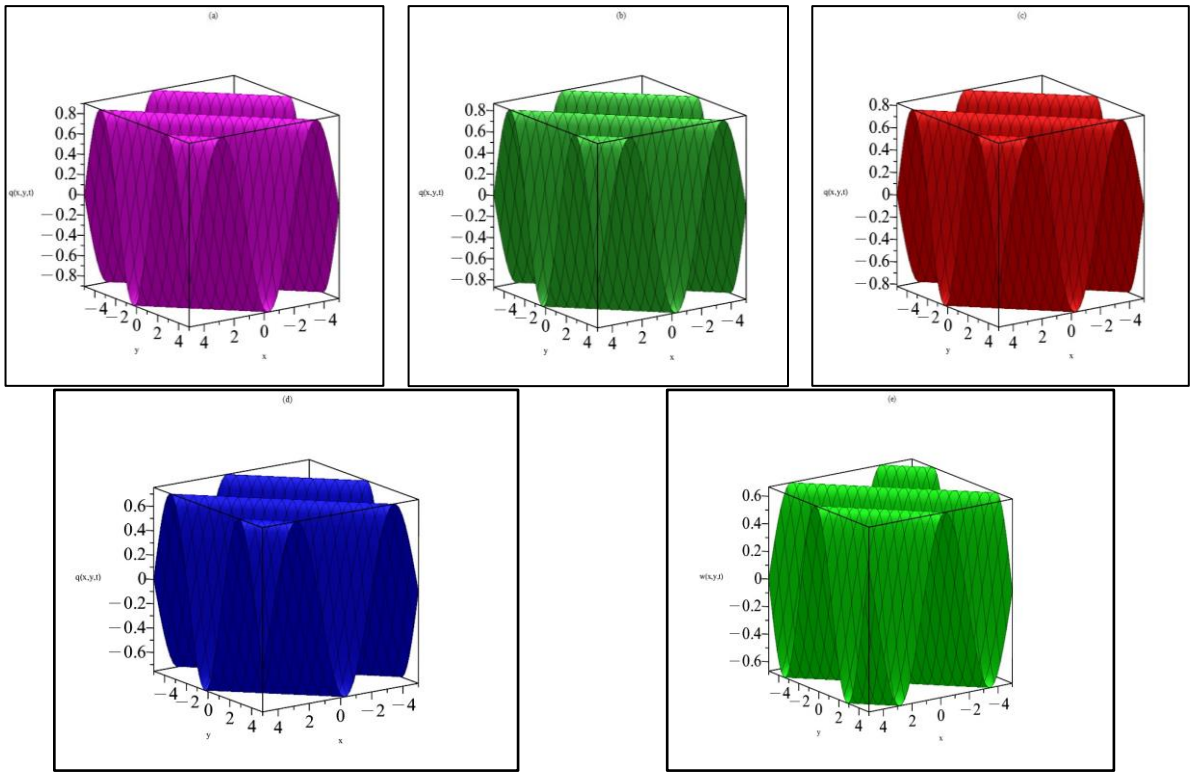


Figure. 2. (a) For $\alpha = 1$, Cq-SHATM solutions (b) For $\alpha = 0.9$, Cq-SHATM solutions (c) For $\alpha = 0.8$, Cq-SHATM solutions (d) For $\alpha = 0.7$, Cq-SHATM solutions (e) For $\alpha = 0.6$, Cq-SHATM solutions at $\rho = 0.5$, $\sigma = 0$, $n = 1$ and $h = -1$ for Equation (28) $h = -1, n = 1$.

In Figure 3, the two dimensional plots for conformable q-Shehu homotopy analysis transform method solution by distinct α values are shown at $x = 0.1, y = 0.1, \rho = 0.5, \sigma = 0,$ $n = 1$ and $h = -1$ for equations (28)-(29).

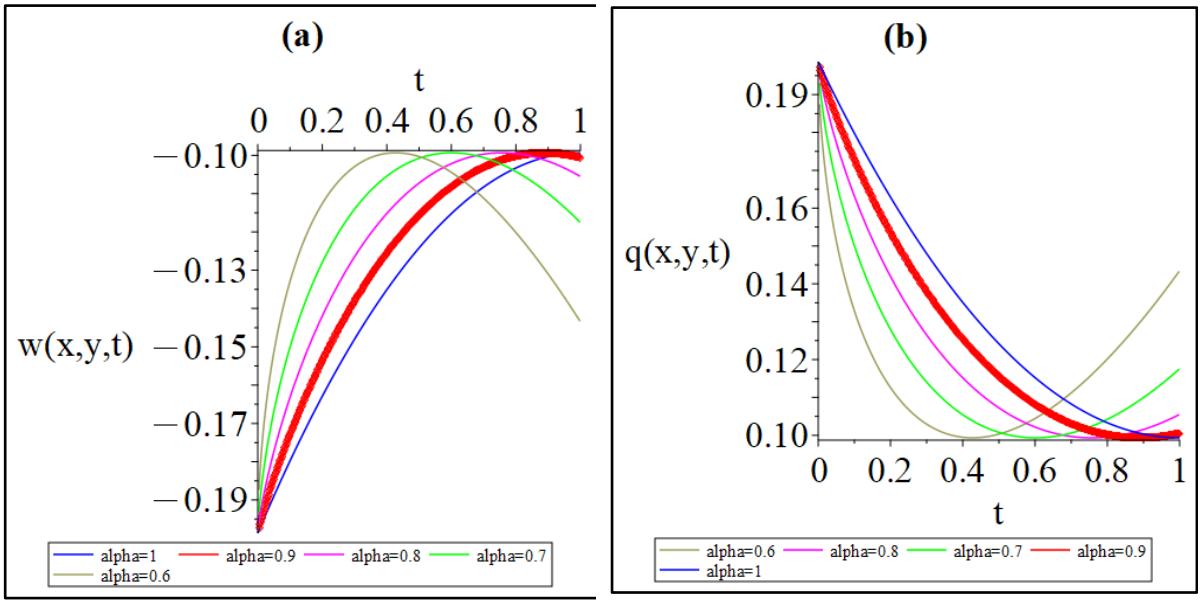


Figure. 3. (a) The two dimensional plot of the Cq-SHATM solutions for equations (28) at $x = 0.1, y = 0.1, \rho = 0.5, \sigma = 0, n = 1$ and $h = -1$ with distinct α values for Eqs.(28). (b) The two dimensional plot of the Cq-SHATM solutions for Eqs. (28) at $x = 0.1, y = 0.1, \rho = 0.5, \sigma = 0, n = 1$ and $h = -1$ with distinct α values for equations (28).

The numerical values of the absolute error between the conformable q-Shehu homotopy analysis transform method solutions and the analytical solution for equation (28) are shown in Table 1.

Table 1. Absolute error between exact solution and approximate solution by Cq-SHATM for equation (28) at $t = 0.01$, $\rho = 0.5, \sigma = 0, n = 1$ and $h = -1$ values.

x/y	0.01	0.02	0.03	0.04	0.05
0.01	3.3×10^{-9}	4.9×10^{-9}	6.6×10^{-9}	8.3×10^{-9}	9.9×10^{-9}
0.02	4.9×10^{-9}	6.6×10^{-9}	8.3×10^{-9}	9.9×10^{-9}	1.1×10^{-8}
0.03	6.6×10^{-9}	8.3×10^{-9}	9.9×10^{-9}	1.1×10^{-8}	1.3×10^{-8}
0.04	8.3×10^{-9}	9.9×10^{-9}	1.1×10^{-8}	1.3×10^{-8}	1.4×10^{-8}
0.05	9.9×10^{-9}	1.1×10^{-8}	1.3×10^{-8}	1.4×10^{-8}	1.6×10^{-8}

The numerical values of the absolute error between the conformable q-Shehu homotopy analysis transform method and the analytical solutions for equation (29) are shown in Table 2.

Table 2. Absolute error between exact solution and approximate solution by Cq-SHATM for equation (29) at $t = 0.01$, $\rho = 0.5$, $\sigma = 0$, $n = 1$ and $h = -1$ values.

x/y	0.01	0.02	0.03	0.04	0.05
0.01	3.3×10^{-9}	4.9×10^{-9}	6.6×10^{-9}	8.3×10^{-9}	9.9×10^{-9}
0.02	4.9×10^{-9}	6.6×10^{-9}	8.3×10^{-9}	9.9×10^{-9}	1.1×10^{-8}
0.03	6.6×10^{-9}	8.3×10^{-9}	9.9×10^{-9}	1.1×10^{-8}	1.3×10^{-8}
0.04	8.3×10^{-9}	9.9×10^{-9}	1.1×10^{-8}	1.3×10^{-8}	1.4×10^{-8}
0.05	9.9×10^{-9}	1.1×10^{-8}	1.3×10^{-8}	1.4×10^{-8}	1.6×10^{-8}

4 RESULTS AND DISCUSSION

Figure 1 indicates the three-dimensional plots of the $w(x, y, t)$ solution via conformable q-Shehu homotopy analysis transform method for different values α of conformable time-fractional Navier-Stokes equation. Figure 2 illustrates the three-dimensional representations of the $q(x, y, t)$ solution via conformable q-Shehu homotopy analysis transform method for different values α of conformable time-fractional Navier-Stokes equation. Figure 3 illustrates the behaviors of conformable q-Shehu homotopy analysis transform method solutions for equations (28)-(29) of conformable time-fractional Navier-Stokes equation, represented by two-dimensional graphs, with varying values of α. Table 1 displays the absolute error between the actual solution and $w(x, y, t)$ derived from the conformable q-Shehu homotopy analysis transform method solution for $\alpha = 1$ in accordance with equation (28). Table 2 presents the absolute error between the exact solution and the $w(x, y, t)$ derived from the conformable q-Shehu homotopy analysis transform method solution for $\alpha = 1$, as indicated in equation (29). Fractional solutions (Figures 1-3) provide new physical insights compared to integer-order models because they capture memory effects and anomalous diffusion more accurately. Unlike integer-order models, which assume local and instantaneous interactions, fractional derivatives introduce a non-local dependency, meaning that the current state of the system is influenced by past states. This is particularly important in fluid dynamics, where real-world flows often exhibit history-dependent behavior. The obtained two dimensional and three dimensional graphical representations in Figures 1-3 illustrate these differences clearly. The fractional-order solutions exhibit smoother transitions and long-range correlations, indicating the presence of memory effects in the fluid's motion. In porous media or non-Newtonian fluids, fractional models better describe subdiffusive or superdiffusive behaviors, whereas integer-order models often oversimplify these effects. The variation in solution behavior with different fractional

orders provides a more flexible framework to model real-world fluids more accurately than classical Navier-Stokes equations. Moreover, error analysis suggests that fractional models achieve higher accuracy in capturing such complex behaviors, reinforcing their significance in practical applications.

5 CONCLUSION AND SUGGESTIONS

The new numerical solutions of conformable time-fractional Navier-Stokes equation have been derived using conformable q-Shehu homotopy analysis transform method. Furthermore, two-dimensional and three-dimensional visualizations of the numerical solutions of conformable time-fractional Navier-Stokes equation have been generated using the Maple software package. The behavior of the numerical solutions for varying α values diverges from the analytical solutions of the equations as the two- and three-dimensional α values move away from unity. Nevertheless, the absolute errors derived from the numerical solutions have proven to be quite small. The derived numerical solutions have shown that conformable q-Shehu homotopy analysis transform method is both effective and reliable. This technique can be used to solve conformable time-fractional partial differential equations numerically.

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Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

A.A.: Conceptualization, Methodology, Validation, Formal Analysis, H.B.: Writing—Original Draft Preparation, and Data Curation, T.A.: Writing—Reviewing and Editing, Visualization, and Investigation.

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